Determining Dynamic Market Equilibrium Price Function Using Second Order Linear Differential Equations

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Abstract

The study of "Determining dynamic market equilibrium price function using second order linear differential equations", aims to determine the equilibrium price function over time in dynamic market equilibrium". The functions obtained by observing the price, price changes, and changes in the level of rising price. Changes in the level of rising price affect the quantity of supply and demand that can be stated in a second order linear differential equation as follows:

$$P'' + \frac{u}{v}P' + \frac{w}{v}P = -\frac{c}{v}$$

The P obtained by adding a particular solution to the homogeneous solution, so that the general solution becomes:

$$1.\ If\ \left(\frac{u}{v}\right)^2 > 4\left(\frac{w}{v}\right),then\quad P(t) = A_1e^{r_1t} + A_2e^{r_2t} - \frac{c}{w}$$

$$2.\ If\ \left(\frac{u}{v}\right)^2 = 4\left(\frac{w}{v}\right),then\quad P(t) = A_1e^{rt} + A_2te^{rt} - \frac{c}{w}$$

$$3.\ If\ \left(\frac{u}{v}\right)^2 < 4\left(\frac{w}{v}\right),then\quad P(t) = e^{at}(A_1\cos(bt) + A_2\sin(bt)) - \frac{c}{w},$$

$$dengan\ a = -\frac{u}{2v}, b = \frac{1}{2}\sqrt{4\left(\frac{w}{v}\right) - \left(\frac{u}{v}\right)^2}$$

Keywords: Second order linear differential equations, dynamic market equilibrium

Introduction

Nowadays, the increasingly of rapid technology has brought changes in our lives, especially in the economic field. The economic issues often encounter in our daily life, such as the rising of grocery prices while demand for goods and services from consumers alone is dwindling. In economic problems will not be separated from the supply and demand for goods and services. It will affect the price stability and will determine the level of prosperity of society in general. Because basically people cannot be separated from the issues of price balance especially the grocery prices. Therefore, this study was conducted to take account of the dynamic market price equilibrium to public in general. Calculations were performed by using the second order linear differential equations method with reference to the number of supply and demand. This method was chosen because this study makes the rising price and changes in the rising price as one of influential factor. The purpose of this study is that people as the parties who have the important role in the market can establish the market equilibrium value in their economy.

Research Methodology

This search is conducted based on the study of literature from various sources, both in the form of a book, or a scientific journal, especially those related to supply, demand, price elasticity and the calculation of market equilibrium price, differential equations, and the of differential equations by using the method of undetermined coefficients. The initial stage is to study the basic concepts of demand, supply, elasticity and the factors that influence the price market equilibrium, and to study about the calculation of the market equilibrium.

The second stage is to study about the concepts of differential equations, second order differential equations as well to study about the general solutions and specific solutions of differential equations by using the method of undetermined coefficients. The last stage is to study the implementation of a second order differential equation in determining the market equilibrium price.

Literature Review

Supply and Demand

Supply is the amount of items sold or offered by the seller in the certain various levels of price and time. Demand is the amount of the desired product and can be purchased by consumers at various price levels and certain period of time. The law adopted in supply is when the price of a good or service increases, the amount of goods or services offered will go up, and if the price decreases, the amount of goods or services offered will go down. While the law adopted in demand is when the price of a good or service increases, the amount of goods or services demanded will go down, and when the price of a good or service goes down then the amount of goods or services demanded will increase[5].

Elasticity

Elasticity is the degree of change in the demand or supply in response to price or income [3]. According to[9], variable elasticity grouped into:

1. (Relatively) Elastic, means that other variables provides a major influence on these variables,
2. (Relatively) Inelastic, means that other variables not cause a major influence on these variables,
3. Unit Elastic, means the percentage change in the variable is equal to the percentage change in the variables that influence,
4. Perfectly Elastic, means that these variables are varies regardless of the value of other variables,
5. Perfectly Inelastic, means that the variable is not influenced by other variables.

In the Elasticity of demand, apart from the price there are a few other variables that affect the demand for, they are[4] :

1. The price of other products that will be a positive influence if the price rises on the substitution product and will be negative on the complementary product.
2. Income Consumers can influence positively but for interior products could have a negative impact.
3. Product Price Expectation in the future influence positively
4. Consumer revenue expectations in the future influence positively
5. Product availability expectations in the future influence negatively
6. Consumer tastesinfluence positively
7. The number of potential consumers influence positively
8. Advertising expenditure influence positivelyand
9. Product attributes are also influence positivelyon demand.

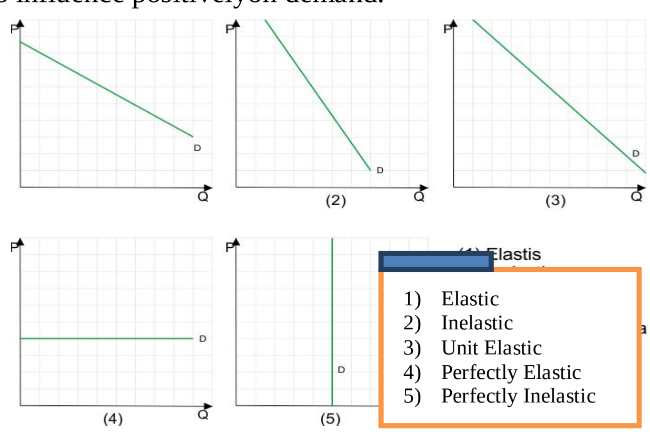

Figure 1: Curve of Variabel Elasticity of the Demand

Market Equilibrium

Market equilibrium achieved when there is no deficiency supply (shortage) or excess supply. Market equilibrium price achieved when the quantity demanded equals to the quantity of goods offered. Thus, to determine the equilibrium price and quantity of market equilibrium, the value of demand function equals to the supply function [5]. There are a lot of supply and demand affect the price of goods at the time, and vice versa. The price of goods when the quantity is at equilibrium is called the equilibrium price. The quantity and the equilibrium price can change over time. It is caused by factors such as trends and economic conditions. A state of equilibrium at a particular time is called static market equilibrium. While the equilibrium and equilibrium changes from time to time is called dynamic market equilibrium.[7].

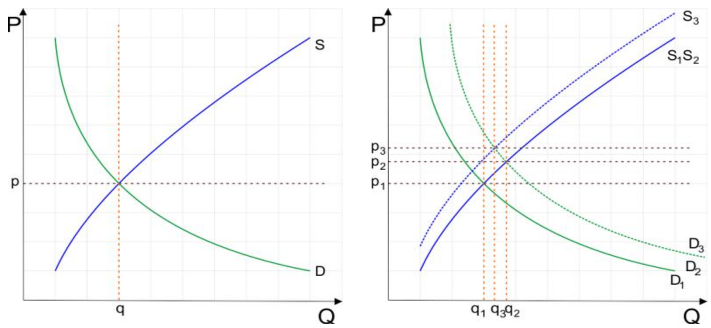

Figure 2: Curve of Static and Dynamic Equilibrium

The figure above is a curve of static equilibrium, where *P* is the price, *Q* is the quantity, *S* is the supply curve, and *D* is the demand curve. On the left curve, *q* is the equilibrium quantity, and *p* is the equilibrium price. On the right curve, S_n, D_n, q_n, $p_n$denote the supply, demand, quantity and price equilibrium at time *n*. In the analysis of supply and demand, the equilibrium can be stated "stable" is when the equilibrium price deviate, it will adjust back to equilibrium prices [8].For example:

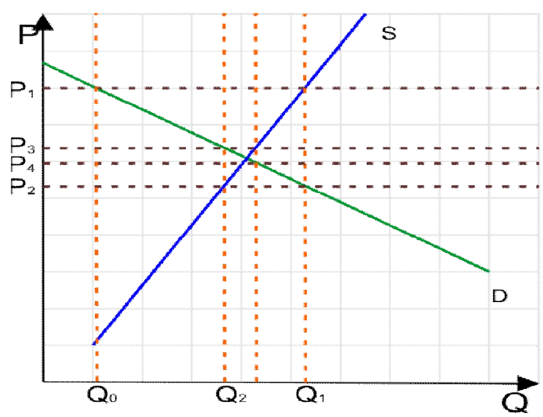

Figure 3. Curve of stable equilibrium

For example, when the price at P_1, a lot of supply; Q1, higher than the demand that is Q_0, due to lack of demand, the price drops to P_2. At a price of P_2, a lot of supply (Q_2) is lower than demand (Q_1), due to the lack of supply, and then the price rises to P_3. At the price of P_3there is over-supply again so the price drops to P_4 and so on so will getting closer to the equilibrium price. The equilibrium is unstable if when there are deviations from the equilibrium price, the price will be further away from the equilibrium price [8].
For example:

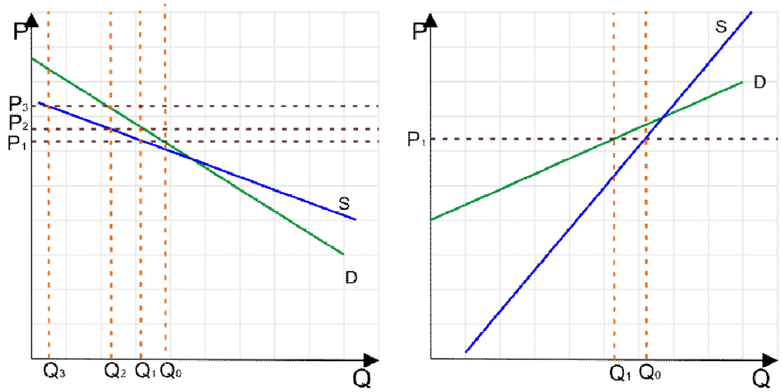

Figure 4: Curve of Unstable Equilibrium

Equilibrium is neutral if when there is a deviation occurs, prices will remain at that value for the equilibrium quantity is unchanged [6].

For example:

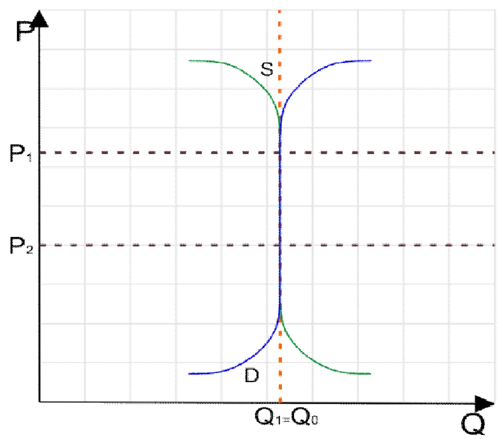

Figure 5: Curve of neutral equilibrium

Result And Discussion

To determine the dynamic market equilibrium price function using second order linear differential equations, look at this figure below:

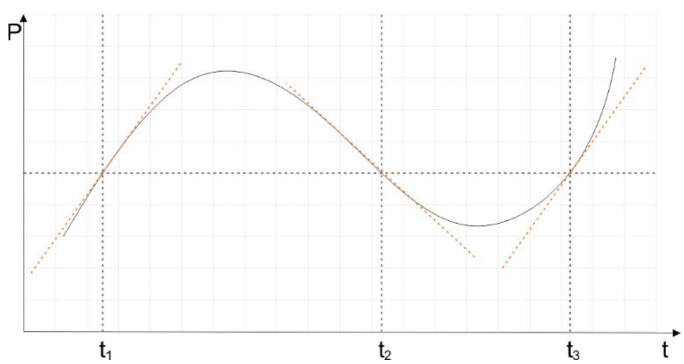

Figure 6: Curve of price function with respect to time Quantity (Q) of demand and supply is affected by the price (P), then the quantity at time t (Q (t)) can be stated by $Q(t)=c+wP(t)$

In the above equation, c is a constant, and w states the influence of price, which has the positive value if the price is directly proportional to the quantity, negative if the price is inversely proportional to the quantity, 0 if the price does not affect the quantity. In the figure 6, the value of $P(t_1)$is equal to $P(t_2)$, but at$t_1$the price is getting higher ($P'(t_1)>0$), while at$t_2$the price is drop ($P'(t_2)<0$). Those price changes is affecting the quantity as well, so $Q(t)$can be stated by $Q(t)=c+wP(t)+uP'(t)$

In the above equation, the price increase is being considered, so there is an addition of *P' (t)* which states the price increase with the coefficient *u which* stating the influence on the quantity. The value of constant *u* can be varying based on the type of goods and other factors. Then, the value of $P(t_1)$is equal to$P(t_3)$, and the value of $P'(t_1)$is equal to $P'(t_3)$, but at$t_1$the price increase is getting lower ($P''(t_1)<0$), while att_3, the price increases drastically increased($P''(t_3)>0$). It affects the quantity as well, so by considering this, *Q (t)*can be stated by

$$Q(t)=c+wP(t)+uP'(t)+vP''(t)$$

In the above equation, the change of price increase is affecting, so there is an addition of *P''(t)*which states the change of price increases level with the coefficient *v* which stating the influence. Thus, the quantity of demand (Q_s) and supply (Q_d) can be $Q_s(t)=c_s+w_sP(t)+u_sP'(t)+v_sP''(t)$

$$Q_d(t)=c_d+w_dP(t)+u_dP'(t)+v_dP''(t) \quad \text{......(1)}$$

The following will be sought the equilibrium price, that is when the quantity of demand and supply are the same($Q_s=Q_d$). From the equation (1), obtained

$$c_s+w_sP(t)+u_sP'(t)+v_sP''(t)=c_d+w_dP(t)+u_dP'(t)+v_dP''(t)$$
$$(w_s - w_d)P(t)+ (u_s - u_d)P'(t)+ (v_s - v_d)P''(t)=-(c_s - c_d) \quad \text{......(2)}$$

By supposing$w=w_s - w_d$, $u=u_s - u_d$, $v=v_s - v_d$, dan $c=c_s - c_d$, equations (2) can be written $wP+ uP'+ vP''= -c$,
Thus, the Dynamic Market Equilibrium Price Function equations is as follow

$$P'' + \frac{u}{v}P' + \frac{w}{v}P = -\frac{c}{v} \quad \text{......(3)}$$

Solving theSecond Order Linear Ordinary Diferensial Equations

Equation (3) is a second order non-homogeneous linear ordinary differential equation. To find a general solution to *P*, will be sought homogeneous solution (P_h) and particulate solution (P_p). P_h obtained by solving the characteristic equation in advance, for this case, the characteristic equation is

$$r^2 + \frac{u}{v}r + \frac{w}{v} = 0$$

$$r = \frac{-\frac{u}{v} \pm \sqrt{(\frac{u}{v})^2 - 4(\frac{w}{v})}}{2} \quad \text{......(4)}$$

For $(u/v)^2>4(w/v)$, the equation has two different roots, the homogeneous solution is

$$P_h = A_1e^{r_1t} + A_2e^{r_2t} \quad \text{......(5)}$$

For $(u/v)^2=4(w/v)$, the equation has two same roots, the homogeneous solution is

$$P_h = A_1e^{rt} + A_2te^{rt} \quad \text{......(6)}$$

For $(u/v)^2<4(w/v)$, the equation has the complex roots, the homogeneous solution is

$$a = -\frac{u}{2v}, b = \frac{1}{2}\sqrt{4\left(\frac{w}{v}\right) - \left(\frac{u}{v}\right)^2}$$

$$P_h = e^{at}(A_1\cos(bt) + A_2\sin(bt)) \quad \text{......(7)}$$

The right side of equation (3) is a constant, so P_p is constant. P_p values obtained by substituting P_p into *P* in equation (3) and finish it.

$$P_p'' + \frac{u}{v}P_p' + \frac{w}{v}P_p = -\frac{c}{v}$$

P_p constant, so $P_p'=0$ and $P_p''=0$

$$0 + \frac{u}{v}(0) + \frac{w}{v}P_p = -\frac{c}{v}$$
$$\frac{w}{v}P_p = -\frac{c}{v}$$
$$P_p = -\frac{c}{w} \quad \text{......(3)}$$

The P obtained by adding a particular solution to the homogeneous solution so that the becomes:

$$1.\ If\ \left(\frac{u}{v}\right)^2 > 4\left(\frac{w}{v}\right),then\quad P(t) = A_1e^{r_1t} + A_2e^{r_2t} - \frac{c}{w}$$

$$2.\ If\ \left(\frac{u}{v}\right)^2 = 4\left(\frac{w}{v}\right),then\quad P(t) = A_1e^{rt} + A_2te^{rt} - \frac{c}{w}$$

$$3.\ If\ \left(\frac{u}{v}\right)^2 < 4\left(\frac{w}{v}\right),then\quad P(t) = e^{at}(A_1\cos(bt) + A_2\sin(bt)) - \frac{c}{w},$$

$$with\ a = -\frac{u}{2v}, b = \frac{1}{2}\sqrt{4\left(\frac{w}{v}\right) - \left(\frac{u}{v}\right)^2}$$

Price Function Over Time

In order to determine the equilibrium price at a certain time, it is required a special solution obtained by finding out the equilibrium price at least 2 times. Then the equilibrium price and the time will be substituted to become an equation system with coefficients A_1 andA_2 as variables. By solving the equations system, it obtained the coefficient values that will be substituted to be a special solution which will be ready to use.

Case example:

Here will be constructed a case example of how to determine the dynamic market equilibrium price function using the second order linear differential equations:

Example, suppose that the supply quantity of goods X when the price of goods at 0 rupiah is 70,000 units. Supply of goods X is increased 1 unit for every 5 rupiah price of these goods. The supply is being added by 9 units of any increase in price of 100 rupiah/month, and decreased 9 units of any price reduction of 100 rupiah/month. The supply also increased by 1 unit if the number of increasing in prices rise by 10 rupiah/month, and decreased 1 unit if the number of increasing price fall to 10 rupiah/month. From this condition, it is obtained the equation of the amount supply of goods X:

$$Q_s(t) = 70000 + \frac{1}{5}P(t) + \frac{9}{100}P^{'}(t) + \frac{1}{10}P^{''}(t)$$
$$= 70000 + 0.2P(t) + 0{,}09P^{'}(t) + 0{,}1P^{''}(t)$$

Similarly, suppose that quantity demanded of goods X when the price of goods at 0 rupiah is 130,000 units. Many demands drop 1 unit every 10 rupiah of the prices of goods X. Demand drops by 3 units for each price rise 100 rupiah/month and rise up 3 units for the price reduction of 100 rupiah/month. Demand is also decrease 1 unit if the number of increasing price rise 10 rupiah/month and increase 1 unit if the number of increasing price drop by 10 rupiah/month from the previous increasing price, and apply for multiplication. From this condition, it can be obtained the equations of amount of demand for goods X:

$$Q_d(t) = 130000 - \frac{1}{10}P(t) - \frac{3}{100}P^{'}(t) - \frac{1}{10}P^{''}(t)$$
$$= 130000 - 0{,}1P(t) - 0{,}03(t) - 0{,}1P^{''}(t)$$

Equilibrium attained when Q_s as much as Q_d so:

$$Q_s(t) = Q_d(t)$$
$$70000 + 0{,}2P(t) + 0{,}09P^{'}(t) + 0{,}1P^{''}(t) = 130000 - 0{,}1P(t) - 0{,}03(t) - 0{,}1P^{''}(t)$$
$$(70000 - 130000) + (0{,}2 + 0{,}1)P(t) + (0{,}09 + 0{,}03)P^{'}(t) + (0{,}1 + 0{,}1)P^{''}(t) = 0$$
$$-60000 + 0{,}3P(t) + 0{,}12P^{'}(t) + 0{,}2P^{''}(t) = 0$$
$$0{,}3P(t) + 0{,}12P^{'}(t) + 0{,}2P^{''}(t) = 60000$$

The equations above is the equation of $wP + uP' + vP'' = -c$ with $w=0,3$, $u=0,12$, $v=0,2$, and $c=-60000$.Then will be compared$(u/v)^2$with$4(w/v)$to find out which solution will be used.

$$\left(\frac{u}{v}\right)^2 = \left(\frac{0{,}12}{0{,}2}\right)^2 = 0{,}36$$
$$4\left(\frac{w}{v}\right) = 4\left(\frac{0{,}3}{0{,}2}\right) = 6$$

Since $(u/v)^2 < 4(w/v)$ then the general solution becomes:

$$P(t) = e^{at}(A_1 \cos(bt) + A_2 \sin(bt)) - \frac{c}{w}$$
$$a = -\frac{u}{2v} = -\frac{0{,}12}{0{,}4} = -0{,}3$$
$$b = \frac{1}{2}\sqrt{4\left(\frac{w}{v}\right) - \left(\frac{u}{v}\right)^2}$$
$$b = \frac{1}{2}\sqrt{6 - 0{,}36}$$
$$b = \sqrt{1{,}41}$$
$$P(t) = e^{-0{,}3t}(A_1 \cos(\sqrt{1{,}41}t) + A_2 \sin(\sqrt{1{,}41}t)) - \frac{-60000}{0{,}3}$$
$$P(t) = e^{-0{,}3t}(A_1 \cos(\sqrt{1{,}41}t) + A_2 \sin(\sqrt{1{,}41}t)) + 200000$$

Example, suppose that the price in January is 186,000 rupiah, and the price in February is 193,000 rupiah, then:

$$P(1) = e^{-0{,}3}(A_1 \cos(\sqrt{1{,}41}) + A_2 \sin(\sqrt{1{,}41})) + 200000$$
$$186000 - 200000 = e^{-0{,}3}(A_1 \cos(\sqrt{1{,}41}) + A_2 \sin(\sqrt{1{,}41}))$$
$$-14000\, e^{-0{,}3} = A_1 \cos(\sqrt{1{,}41}) + A_2 \sin(\sqrt{1{,}41})$$
$$P(2) = e^{-0{,}6}(A_1 \cos(2\sqrt{1{,}41}) + A_2 \sin(2\sqrt{1{,}41})) + 200000$$
$$193000 - 200000 = e^{-0{,}6}(A_1 \cos(2\sqrt{1{,}41}) + A_2 \sin(2\sqrt{1{,}41}))$$
$$-7000\, e^{-0{,}6} = A_1 \cos(2\sqrt{1{,}41}) + A_2 \sin(2\sqrt{1{,}41})$$

By solving the equation system above, it can be obtained:

$$A_1 = -1382{,}426132$$
$$A_2 = -19819{,}59861$$

By substituting A_1 and A_2 to the general solution, it is obtained the particular solutions:

$$P(t) = e^{-0{,}3t}(-1382{,}426132 \cos(\sqrt{1{,}41}t) + -19819{,}59861 \sin(\sqrt{1{,}41}t)) + 200000$$

Function Graph P (t) of the price of goods X over time t = 36 months are shown in figure (7)

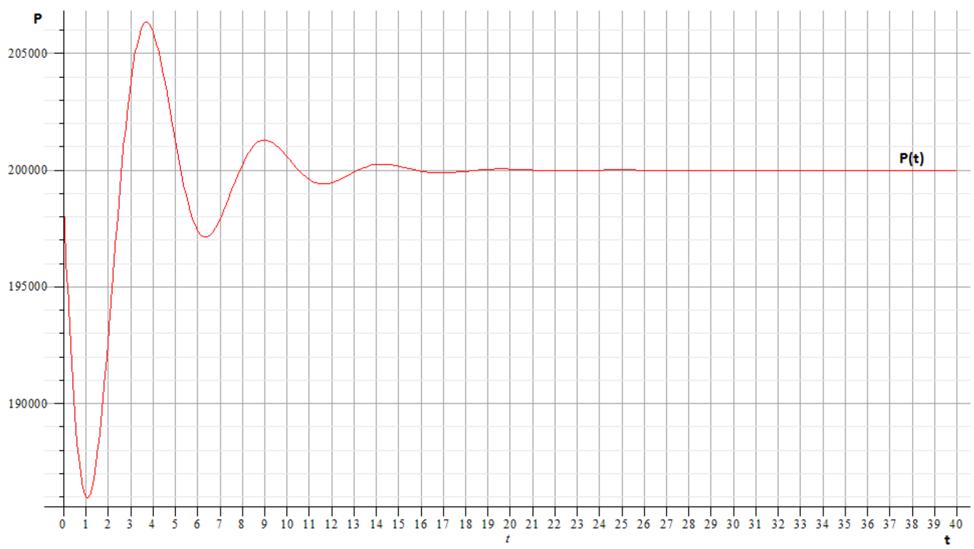

Table 1: Table of equilibrium price of goods X during t = 36 months

Month	Equilibrium Price (Rp)
1	186,000
2	197,000
3	203,804
4	205,949
5	201,209
6	197,405
7	197,898
8	200,259
9	201,297
10	200,576
11	199,607
12	199,466
14	200,248
16	199,964
18	199,953
20	200,047
22	199,976
24	200,004
30	200,002
36	200,000

From the figure (7) and table 1, it can be seen that the longer, the price of goods will stabilize close to the value of 200,000 rupiah. The following will be tested whether the prices above are the equilibrium price on each of the month. Testing is done by looking for value *P' (t)*and *P''(t)*, then those are used to find the equilibrium quantity of supply *(Qs)* and demand *(Qd)*.

$$\begin{aligned} P'(t) &= -0{,}3\mathrm{e}^{-0{,}3t}(-1382{,}426126\cos(1{,}187434209t) - 19819{,}59861\sin(1{,}187434209t)) \\ &\quad + \mathrm{e}^{-0{,}3t}(-23534{,}46940\cos(1{,}187434209t) + 1641{,}540073\sin(1{,}187434209t)) \end{aligned}$$

$$\begin{aligned} P''(t) &= 0{,}09e^{-0{,}3t}\big(-1382{,}426126\cos(1{,}187434209t) - 19819{,}59861\, sin(1{,}187434209t)\big) \\ &\quad - 0{,}6e^{-0{,}3t}\big(-23534{,}46940\, cos(1{,}187434209t) + 1641{,}540073\, sin(1{,}187434209t)\big) \\ &\quad + e^{-0{,}3t}(27945{,}63406\, sin(1{,}187434209t) + 1949.220838\, cos(1.187434209t)) \end{aligned}$$

Table 2: Comparison table *$Q_s(t)$* and *$Q_d(t)$*

Bulan (t)	P'(t)	P''(t)	Q_s(t)	Q_d(t)
1	-1193.50	21716.10	109264	109264
2	12026.95	3283.83	110011	110011
3	7320.25	-10098.18	110410	110410
4	-2543.71	-7398.53	110221	110221
5	-5427.14	1441.79	109898	109898
6	-1611.66	4859.43	109822	109822
7	2085.31	1901.79	109957	109957
8	2040.16	-1612.95	110074	110074
9	-13.80	-1937.61	110064	110064
10	-1127.31	-188.60	109995	109995

From the table 2, it can be seen that there are many demands are equal to the supply. This result shows that the price (*P (t)*), the price increase (*P'(t)*), and the big changes in the level of rising price(*P''(t)*) which had been obtained is in conformity as expected.

Conclusion

The function of the equilibrium price over time can be obtained by considering the price, price changes, and the changes in the level of rising price by assuming that no other factors that affect the equilibrium.

The functions which had been obtained need parameters stating the affect of price, price changes, and changes in the rising price. Those parameters are obtained based on assumptions or observations of phenomena that occur in a dynamic market.

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